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Cohomology of Schematic Algebras

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Let R be a positively graded Noetherian k-algebra and let Proj $R = (R, \kappa_+) - gr$ be the quotient category of the category of graded R-modules $R - gr$ modulo those of finite length. If R is schematic (cf. [F. Van Oystaeyen and L. Willaert, *J. Pure Appl. Algebra* **104** (1995), 109–122]) and has finite global dimension, we show that the cohomology groups of R (as defined in [M. Artin and J. J. Zhang, *Adv. in Math.* **109** (1994), 228–287]) are finite dimensional. Furthermore, we introduce Čech cohomology groups for schematic algebras and prove that they coincide with the usual graded cohomology groups. These generalized Čech cohomology groups are still manageable; we illustrate this with an example.

In [10], schematic algebras were introduced as positively graded algebras possessing ‘‘enough’’ Ore sets. They admit a scheme theory which is remarkably similar to the scheme theory of a commutative algebra. All commutative algebras are schematic, but the class of schematic algebras contains many more interesting examples (cf. [9]). For commutative rings, scheme theory may be applied in order to get a practical method for computing cohomology groups in $(R, \kappa_+) - gr$, the quotient category of the graded R-modules modulo those of finite length. Indeed, cohomology in the quotient category may be calculated in the category of sheaves, and the cohomology groups of quasi-coherent sheaves in the latter category coincide with the Čech cohomology groups. The graded cohomology groups also make sense for a non-commutative connected k-algebra (cf. [2]). This paper tries to answer the natural question whether the process we have described above applies to all schematic algebras.

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In the first two sections, we provide a brief introduction to schematic algebras and cohomology groups in $(R, \kappa_+) - gr$. The reader is referred to [10, 2] for more detail. We also show that the cohomology groups of schematic algebras with finite global dimension are finite dimensional. In the third section, we develop Čech cohomology for schematic algebras. We prove that for a graded injective R-module the Čech cohomology groups vanish. It follows easily that the Čech cohomology groups coincide with the graded cohomology groups of $(R, \kappa_+) - gr$. In the last section, we illustrate the usefulness of this theorem by calculating some cohomology groups which play an important role in [4].

1. PRELIMINARIES

Let k denote a field and let R be a Noetherian connected k-algebra. This means that R is positively graded and that its part of degree 0 is equal to k: $R = k \oplus R_+$ where $R_+ = \oplus_{n \geq 1} R_n$. The category of graded R-modules will be denoted by $R - gr$. The set $\{(R_+)^n : n \in \mathbb{N}\}$ is an idempotent filter (cf. [7]). Hence we have a localization functor Q_{κ_+} available. Moreover, the corresponding kernel functor κ_+ (the one which sends a graded R-module to its torsion submodule: $\kappa_+(M) = \{m \in M \mid \exists n \in \mathbb{N} : (R_+)^n m = 0\}$) is stable, i.e., the injective envelope of a torsion module is again torsion. The quotient category $(R, \kappa_+) - gr$ consists of all κ_+-*closed* graded R-modules, i.e., those graded R-modules which satisfy $M \cong \mathrm{Hom}_R((R_+)^n, M)$ for all n. More details about the quotient category may be found in [7] or in [5], but we will provide an alternative description in the next section.

The notion ‘‘schematic algebra’’ has been introduced in order to provide a scheme-like description of the quotient category $(R, \kappa_+) - gr$ (cf. [10], [9]). The definition goes as follows:

DEFINITION. R is *schematic* if there exists a finite number of two-sided homogeneous Ore sets $S_1, \ldots, S_N$ such that for all $i \in \{1, \ldots, N\}$: $S_i \cap R_+ \neq \emptyset$ and such that for all $(s_i)_{i=1,\ldots,N} \in \prod_{i=1}^N S_i$, $\exists m \in \mathbb{N}$ with $(R_+)^m \subseteq \sum_{i=1}^N R s_i$.

Let $\mathscr{W}$ be the free monoid on all two-sided Ore sets of R. This set is used in [10] in order to define a kind of categorical topology: if $V, W \in \mathscr{W}$, then we have an ‘‘inclusion’’ map $V \to W$ if we meet all letters of W (in the right order) while reading V. A global cover is defined as a finite subset $\{W_i : i \in I\}$ of $\mathscr{W}$ such that all graded R-modules with $Q_{W_i}(M) = 0$ for all i must have $Q_{\kappa_+}(M) = 0$. (If a word W is equal to $S_1, \ldots, S_n$, then $Q_W(M)$ denotes $Q_{S_n}(R) \otimes_R \cdots Q_{S_1}(R) \otimes_R M$.) A presheaf on this topological space is just a contravariant functor $\mathscr{F}$ from the category $\mathscr{W}$ associated to $\mathscr{W}$ to the category of graded R-modules. In particular, each "inclusion" $V \to W$ induces a graded morphism $\rho_V^W : \mathscr{F}(W) \to \mathscr{F}(V)$. Such a presheaf is a sheaf if it satisfies the usual separatedness and gluing conditions. Every graded R-module M induces a sheaf $\tilde{M}$ on this topology with global sections precisely $Q_{\kappa_+}(M)$ and with sections on a word W isomorphic to $Q_W(M)$. Quasicoherent sheaves may be defined as sheaves which are locally isomorphic to a sheaf stemming from a module. The main result of [10] states that Serre's Theorem still holds: the category of quasi-coherent sheaves is equivalent to the quotient category $(R, \kappa_+) - gr$.

The aim of this paper is a study of the cohomology groups of a schematic algebra R. These groups have been defined in [2]; in the next section we provide a brief introduction.

2. COHOMOLOGY IN THE QUOTIENT CATEGORY

Let R be any Noetherian connected k-algebra. Define a category $\mathscr{C}$ with the same objects as $R - gr$, the category of graded R-modules. We will write $\pi(M)$ when considering the graded R-module M as an object of $\mathscr{C}$. Morphisms in $\mathscr{C}$ are defined as follows,

$$\operatorname{Hom}_{\mathscr{C}}(\pi(M), \pi(N)) = \varinjlim_{M'} \operatorname{Hom}_{R-gr}\big(M', N/\kappa_+(N)\big),$$

where M' runs over the category of submodules of M such that M/M' is torsion. Consequently, π is an exact functor from $R - gr$ to $\mathscr{C}$. Moreover, π has a right adjoint $\omega : \mathscr{C} \to R - gr$, in the sense that

$$\operatorname{Hom}_{R-gr}(M, \omega(\mathscr{N})) \cong \operatorname{Hom}_{\mathscr{C}}(\pi(M), \mathscr{N}).$$

These two functors establish an equivalence between $\mathscr{C}$ and the quotient category $(R, \kappa_+) - gr$. Since $\mathscr{C}$ has enough injectives, we may define H^i, the ith right derived functor of $\operatorname{Hom}_{\mathscr{C}}(\pi(R), -)$. In order to calculate $H^i(\pi(M))$, we should start with an injective resolution of $\pi(M)$ in $\mathscr{C}$, apply the functor $\operatorname{Hom}_{\mathscr{C}}(\pi(R), -)$, and take homology on the ith place. We get an injective resolution of $\pi(M)$ in $\mathscr{C}$ if we apply the functor π to an injective resolution $E^{\cdot}$ of M in $R - gr$. Moreover, since

$$\operatorname{Hom}_{\mathscr{C}}\big(\pi(R), \pi(E^i)\big) \cong \operatorname{Hom}_{R-gr}\big(R, \omega\pi(E^i)\big) \cong \big(\omega\pi(E^i)\big)_0$$

we get that $H^i(\pi(M)) \cong h^i(\omega\pi(E^{\cdot})_0)\ \forall i \in \mathbb{N}$. If one defines the shifted module $M[n]$ as the module M with gradation $(M[n])_p = M_{n+p}$, one obtains the graded cohomology groups by:

$$\mathbf{H}^i(\pi(M)) \overset{\text{def}}{=} \bigoplus_{n \in \mathbb{Z}} H^i(\pi(M[n])).$$

In particular, $\mathbf{H}^0(\pi(M)) \cong \omega\pi(M) \cong Q_{\kappa_+}(M)$. These graded cohomology groups are again graded R-modules and from the reasoning above we obtain that

$$\mathbf{H}^i(\pi(M)) \cong h^i(\omega\pi(E^{\cdot})).$$

Assume from now on that R is schematic. The next section of this paper is devoted to an alternative description of these graded cohomology groups $\mathbf{H}^i(\pi(M))$. We will now derive some of their properties. This task has already been undertaken in [2]. The authors of [2] show that $\mathbf{H}^i(\pi(M))$ is a torsion module for all $i > 0$ and that further properties may be obtained if the algebra R satisfies a rather technical condition, named χ. Schematic algebras with finite global dimension do satisfy this condition: the following lemma, phrased in the *category* $R - \text{mod}$ *of an arbitrary Noetherian ring* R, is useful to this end:

LEMMA 1. *If S is a two-sided Ore set in a Noetherian ring R and M is a finitely generated left R-module then*

$$Q_S\big(\mathrm{Ext}^n_R({}_RM, {}_RR)\big) \cong \mathrm{Ext}^n_{Q_S(R)}\big(Q_S(M), Q_S(R)\big).$$

A proof may be found in [3, Proposition 1.6].

THEOREM 1. *If R is a schematic algebra having finite global dimension, then R satisfies condition χ of* [2].

Proof. Let M be a graded left R-module. Then the k-vectorspace $\mathrm{Ext}^n_R({}_Rk, {}_RM)$ obtains a left R-module structure since k is also a right R-module. We have to prove that this graded left R-module is bounded (i.e., finite dimensional) for each finitely generated graded left R-module M. It is easy to see that $\mathrm{Ext}^n_R({}_Rk, {}_RM)$ is torsion. Instead of proving that it is also finitely generated, we consider $\mathrm{Ext}^n({}_Rk, {}_RR)$. Due to the bimodule structure of R, this left R-module is also a right R-module. If $S_1, \dots, S_n$ is a finite cover of R, then applying the lemma yields:

$$Q_{S_i}\big(\mathrm{Ext}^n_R({}_Rk, {}_RR)\big) \cong \mathrm{Ext}^n_{Q_{S_i}(R)}\big(Q_{S_i}({}_Rk), Q_{S_i}({}_RR)\big).$$

As the left localization of ${}_Rk$ at S_i is zero, it follows that the right localization of $\mathrm{Ext}^n_R({}_Rk, {}_RR)$ at each S_i is zero; hence $\mathrm{Ext}^n_R({}_Rk, {}_RR)$ is torsion as a right R-module. On the other hand, considering a resolution of ${}_Rk$ by sums of shifts of R shows that $\mathrm{Ext}^n_R({}_Rk, {}_RR)$ is a finitely generated right R-module. Combining these two facts yields that it is finite dimensional. Consider a finitely generated graded projective left R-module P. Then P is a finite sum of shifts of R (see [1]), so $\mathrm{Ext}^n({}_Rk, {}_RP)$ is also finite dimensional. Finally, induction on the projective dimension yields that

$\mathrm{Ext}_R^n({}_Rk, {}_RM)$ is finite dimensional for each finitely generated graded left R-module M. In the terminology of [2], this means that R satisfies χ^0 and hence χ since R is locally finite. ∎

The next consequences of Theorem 1 now follow immediately from results in [2]:

COROLLARY 1. *If R is a schematic algebra with finite global dimension, then for each finitely generated graded left R-module M, we have*:

1. $Q_{\kappa_+}(M) \cong \mathrm{Hom}_R(R_{\geq n}, M)$ *if n is sufficiently large*.
2. $Q_{\kappa_+}(M)_{\geq n} \cong M_{\geq n}$ *if n is sufficiently large*.
3. $Q_{\kappa_+}(M)_{\geq n}$ *is a finitely generated R-module for all n*.
4. $\forall j \geq 1 : (\mathbf{H}^j(\pi(M)))_n$ *is finite dimensional for all n*.
5. $\forall j \geq 1 : (\mathbf{H}^j(\pi(M)))_n = 0$ *if n is sufficiently large*.
6. $\mathrm{Ext}_R^n(N, M)$ *is finite dimensional for all n if N is a finite-dimensional R-module*.

Since a connected graded algebra with finite global dimension satisfying χ is Artin–Schelter regular [11], we obtain:

COROLLARY 2. *A schematic algebra R with finite global dimension d is Artin–Schelter regular, i.e.*, $\mathrm{Ext}_R^i(k, R) = 0$ *for $i \neq d$ and* $\mathrm{Ext}_R^d(k, R) = k[n]$ *for some integer n*.

3. ČECH COHOMOLOGY

The sheaf-like description of $(R, \kappa_+) - gr$ for a schematic algebra R admits a Čech cohomology theory which is much the same as in the commutative case. Again, the aim is providing a more practical way to calculate the graded cohomology groups $\mathbf{H}^i(\pi(M))$. We assume that R is a Noetherian connected k-algebra and is moreover schematic.

Start with a finite global cover $\mathscr{W} = (W_i)_{i \in I}$. For any $\sigma = (i_0, \ldots, i_p) \in I^{p+1}$ we define $W_\sigma = W_{i_0} W_{i_1} \cdots W_{i_p}$. If $\mathscr{F}$ is a sheaf on $\mathscr{W}$, then we define a complex $C^p(\mathscr{W}, \mathscr{F})$ of graded R-modules as follows:

$$C^p(\mathscr{W}, \mathscr{F}) = \prod_{\sigma \in I^{p+1}} \Gamma(W_\sigma, \mathscr{F}).$$

As usual, $\Gamma(W,\mathscr{F}) = \mathscr{F}(W)$ denotes the sections of the sheaf $\mathscr{F}$ on the "open set" W. Thus an element $s \in C^p(\mathscr{W},\mathscr{F})$ is determined by elements

$$s_{i_0\dots i_p} \in \mathscr{F}\big(W_{i_0}\dots W_{i_p}\big)$$

for each $p+1$-tuple $(i_0,\dots,i_p)$ of I^{p+1}. For each $j \in \{0,\dots,p+1\}$, we consider the map $\partial_j : I^{p+2} \to I^{p+1}$ which sends $(i_0,\dots,i_{p+1})$ to $(i_0,\dots,\hat{i}_j,\dots,i_{p+1})$, where $\hat{i}_j$ means that this index is omitted. The coboundary-map $d_p : C^p(\mathscr{W},\mathscr{F}) \to C^{p+1}(\mathscr{W},\mathscr{F})$ is now defined by the following rule:

$$\big(d_p s\big)_\sigma = \sum_{j=0}^{p+1} (-1)^j \rho_{W_\sigma}^{W_{\partial_j(\sigma)}}\big(s_{\partial_j(\sigma)}\big) \qquad \text{for each } \sigma = \big(i_0,\dots,i_{p+1}\big) \in I^{p+2}.$$

As usual, we have that $d_{p+1}\circ d_p = 0$, so we may define $\check{H}^p(\mathscr{W},\mathscr{F})$, the pth Čech cohomology group of $\mathscr{F}$ with respect to the covering $\mathscr{W}$, as the pth homology group of this complex. It follows from the very definition of sheaves that $\check{H}^0(\mathscr{W},\mathscr{F}) \cong \Gamma_*(\mathscr{F})$, the global sections of $\mathscr{F}$.

We would like to prove that for all graded R-modules M, the Čech cohomology groups $\check{H}^i(\mathscr{W},\tilde{M})$ are isomorphic to the graded cohomology groups $\mathbf{H}^i(\pi(M))$. Therefore, we should study the localizations of graded injective R-modules. Call a module *flasque* if for all words W the "localization" map $M \to Q_W(M)$ is surjective (or equivalently, if all restriction maps of the sheaf $\tilde{M}$ are surjective). A generalization of the fact that every injective module is divisible (cf. [6, Theorem 3.23]) yields:

THEOREM 2. *If E is injective in $R-gr$, then E is flasque.*

Proof. We prove by induction on the length of the word W that the map $E \to Q_W(E)$ is surjective. If W is an Ore set S, then we consider a homogeneous element n/s of $Q_S(E)$. Without loss of generality, we may assume that $\deg n = \deg s$. The ascending chain $\mathrm{Ann}(s) \subseteq \mathrm{Ann}(s^2) \subseteq \dots$ becomes stationary (since R is supposed to be Noetherian). Say $\mathrm{Ann}(s^r) = \mathrm{Ann}(s^{r+n})$ $\forall n \in \mathbb{N}$. Consequently, the map ψ: $Rs^{r+1} \to E$ with $\psi(s^{r+1}) = s^r n$ is a well-defined graded R-module homomorphism and hence there is a $\varphi \in \mathrm{Hom}_{R-gr}(R,E)$ such that φ coincides with ψ on Rs^{r+1}. If $z = \varphi(1)$, then it is easy to see that $z/1 = n/s \in Q_S(E)$. As to the general case, if we write $W = SV$, then $E \to Q_W(E)$ is equal to the composition $E \to Q_V(E) \to Q_V(Q_S(E))$, the first one being surjective by induction, the second one due to the surjectivity of $E \to Q_S(E)$ and the exactness of the functor Q_V. ∎

COROLLARY 3. *If E is injective in $R - gr$, then $\forall W \in \mathscr{W}$.*

$$0 \to \kappa_W(E) \to E \to Q_W(E) \to 0$$

is exact in $R - gr$.

THEOREM 3. *If E is injective in $R - gr$, then $\check{H}^i(\mathscr{W}, \tilde{E}) = 0$ for all $i > 0$.*

Proof. Fix a (finite) global cover $\mathscr{W} = (W_i)_{i \in I}$. Since R is graded Noetherian, the category $R - gr$ possesses a set of Noetherian generators, namely $\{R[n] : n \in \mathbb{Z}\}$. Hence, $R - gr$ is a "locally Noetherian Grothendieck category" and we may apply Theorem 8.11 of [5] in order to conclude that each graded injective may be written as a direct sum of directly indecomposable graded injectives. As the functors $\check{H}^i$ commute with direct sums, it suffices to prove that $\check{H}^i(\tilde{E}) = 0$ if E is graded directly indecomposable injective. But κ_+ is stable and hence $\kappa_+(E)$ is a direct summand of E in $R - gr$, yielding that $\kappa_+(E)$ is either E or either 0. In the first case, it is trivial that $\check{H}^i(\tilde{E}) = 0$ for all $i \geq 0$. Hence we may suppose we are in the second case, i.e., that E is moreover torsionfree.

If it were true that for all $i \in I : \kappa_{W_i}(E)$ is non-zero, then we would be able to choose a non-zero homogeneous element e_i of $\kappa_{W_i}(E)$. On one hand, the intersection $\bigcap_i Re_i$ should be non-zero, since each Re_i is an essential subobject of E in $R - gr$. On the other hand, $\bigcap_i Re_i \subseteq \bigcap_i \kappa_{W_i}(E) = \kappa_+(E) = 0$, a contradiction. We conclude that there is at least one $i \in I$ with $\kappa_{W_i}(E) = 0$. Consequently, $Q_{W_i}(E) = E$ and $Q_{W_i W}(E) = Q_W(E)$. It follows that the maps

$$h_p : C^p(\mathscr{W}, \tilde{E}) \to C^{p-1}(\mathscr{W}, \tilde{E}) \qquad (p \geq 1)$$

given by $h_p(s)_\tau = s_{(i,\tau)} \in Q_{W_i W_\tau}(E) = Q_{W_\tau}(E)$ $(\tau \in I^p)$ are well defined. If we are able to show that these maps form a contracting homotopy, then it follows that the Čech complex is exact and hence $\check{H}^i(\mathscr{W}, \tilde{E}) = 0$ for all $i > 0$.

So take an element $s = (s_\sigma)_{\sigma \in I^{p+1}} \in C^p(\mathscr{W}, \tilde{E})$. We have to show that $d_{p-1} \circ h_p(s) + h_{p+1} \circ d_p(s) = s$. Fix a σ in I^{p+1} and calculate:

$$\begin{aligned} \big(d_{p-1} \circ h_p(s)\big)_\sigma &= \sum_{j=0}^{p} (-1)^j \rho_{W_\sigma}^{W_{\partial_j(\sigma)}}\big(h_p(s)_{\partial_j(\sigma)}\big) \\ &= \sum_{j=0}^{p} (-1)^j \rho_{W_\sigma}^{W_{\partial_j(\sigma)}}\big(s_{(i,\partial_j(\sigma))}\big). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\big(h_{p+1}\circ d_p(s)\big)_\sigma &= d_p(s)_{(i,\sigma)}\\
&= \sum_{j=0}^{p+1}(-1)^j \rho^{W_{\partial_j(i,\sigma)}}_{W_{(i,\sigma)}}\big(s_{\partial_j(i,\sigma)}\big)\\
&= \rho^{W_\sigma}_{W_i W_\sigma}(s_\sigma) + \sum_{j=1}^{p+1}(-1)^j \rho^{W_{(i,\partial_{j-1}(\sigma))}}_{W_{(i,\sigma)}}\big(s_{(i,\partial_{j-1}(\sigma))}\big)\\
&= s_\sigma + \sum_{j=0}^{p}(-1)^{j-1}\rho^{W_{\partial_j(\sigma)}}_{W_\sigma}\big(s_{(i,\partial_j(\sigma))}\big)\\
&= s_\sigma - \big(d_{p-1}\circ h_p(s)\big)_\sigma. \quad \blacksquare
\end{aligned}
$$

The desired result is now a standard consequence of the previous theorem:

THEOREM 4. *For all graded R-modules M, for all finite global covers $\mathscr{W}$, we have that* $\mathbf{H}^i(\pi(M)) \cong \check{H}^i(\mathscr{W}, \tilde{M})$.

Proof. If $0 \to M \to N \to P \to 0$ is an exact sequence in $R - gr$, then from the exactness of the functors Q_W $(W \neq 1)$, we obtain for each $p \geq 0$ an exact sequence $0 \to C^p(\mathscr{W}, \tilde{M}) \to C^p(\mathscr{W}, \tilde{N}) \to C^p(\mathscr{W}, \tilde{P}) \to 0$. It follows that the homology functors $\check{H}^i$ form a so-called exact ∂-functor (cf. [8, Lemma 2.12]). Moreover, the previous theorem shows that this ∂-functor is effaceable. It is well known (cf. [8, Theorem 2.13]) that these two facts imply that the functor $\check{H}^i$ is equivalent to the ith derived functor of $Q_{\kappa_+} : R - gr \to R - gr$ and the latter is precisely $\mathbf{H}^i \circ \pi$. ▮

If R is a connected graded algebra which is a finite module over its Noetherian center $Z(R)$, then R is schematic (use the Ore sets consisting of the powers of the generators of $Z(R)$). The Čech complex we use is then exactly the Čech complex for graded $Z(R)$-modules as defined in [8]. Consequently, if M is a graded R-module, then $\mathbf{H}^i_R(\pi(M))$, considered as a $Z(R)$-module, is just $\mathbf{H}^i_{Z(R)}(\pi(M))$! Consequences of this are being worked out.

We conclude this section with a few words on sheaf cohomology. Let *Psh* denote the category of all presheaves of abelian groups on the non-commutative site $\mathscr{W}$, i.e., all contravariant functors $\mathscr{W} \to Ab$. A presheaf is called a sheaf if it satisfies the usual conditions. Sheaves on $\mathscr{W}$ form a full subcategory *Sh* of *Psh*. The natural inclusion functor $i : Sh \to Psh$ is left exact and has a left adjoint $a : Psh \to Sh$. It follows that the categories *Psh* and *Sh* have enough injectives. Hence we may define the

ith right derived functor $\mathscr{H}^i$ of the global sections functor $\Gamma_* : Sh \to Ab$. We have not been able to show that for a coherent sheaf these sheaf cohomology groups coincide with the graded cohomology groups of its global sections (or, equivalently, with its Čech cohomology groups):

QUESTION. *Is it true that for all graded R-modules $M : \mathbf{H}^i(\pi(M)) \cong \mathscr{H}^i(\tilde{M})\ \forall i \in \mathbb{N}$?*

4. AN EXAMPLE

Let H be the homogenization of the first Weyl algebra, i.e., H is the graded algebra generated by three homogeneous elements X, Y, and Z of degree 1 which satisfy the relations:

$$\begin{aligned} XY - YX &= Z^2 \\ XZ - ZX &= 0 \\ YZ - ZY &= 0. \end{aligned}$$

This algebra has excellent properties: it is an Auslander regular domain of global dimension 3 and it satisfies the Cohen−Macaulay property. Using the fact that H has a basis consisting of the ordered monomials $X^iY^jZ^k$ $(i, j, k \geq 0)$, it is easy to see that the Ore sets $\{X^n | n \in \mathbb{N}\}$, $\{Y^n | n \in \mathbb{N}\}$, and $\{Z^n | n \in \mathbb{N}\}$ make H schematic.

In [4], cohomology groups are used in order to classify the right ideals of the Weyl algebra $A = A_1(k)$. Consider the homogenizations of the "canonical" non-principal right ideals of $A : P_n = X^{n+1}H + (XY + nZ^2)H$ $(n \in \mathbb{N})$. It is shown in [4] that P_n is determined by the vectorspaces $H^1(\pi(P_n[-1]))$ and $H^1(\pi(P_n))$ and the matrices of the multiplications by X, Y, and Z.

We will now show that the methods introduced in the foregoing section are still as useful as in the commutative case by calculating the entire cohomology groups $\mathbf{H}^1(\pi(P_n))$. Fix $n \in \mathbb{N}$ and let $P = P_n$. The quotient H/P will be denoted by M and the equivalence classes of X, Y, and Z will be denoted by x, y, and z. We will calculate the Čech cohomology of M and use the long exact cohomology sequence in order to determine $\mathbf{H}^i(\pi(P))$. This is advantageous since M has a lot of torsion: M is clearly X-torsion and the equality $(n > m)$

$$X^nY^{2m} = Y^mX^{n-m} \prod_{i=0}^{m-1} \left(XY + (n - m + 1 + 3i)Z^2\right)$$

entails that $Q_Y(M)$ is also X-torsion. Whence

$$
\begin{aligned}
\mathscr{C}^0(\pi(M)) &= Q_Y(M) \oplus Q_Z(M)\\
\mathscr{C}^1(\pi(M)) &= Q_{YY}(M) \oplus Q_{YZ}(M) \oplus Q_{ZY}(M) \oplus Q_{ZZ}(M)\\
\mathscr{C}^2(\pi(M)) &= Q_{YYY}(M) \oplus Q_{YYZ}(M) \oplus Q_{YZY}(M) \oplus Q_{YZZ}(M)\\
&\qquad \oplus Q_{ZYY}(M) \oplus Q_{ZYZ}(M) \oplus Q_{ZZY}(M) \oplus Q_{ZZZ}(M)
\end{aligned}
$$

and so on. Moreover, the functors Q_Y and Q_Z commute, so the Čech complex is completely built with the three modules $Q_Y(M)$, $Q_Z(M)$, and $Q_{YZ}(M)$. Since M is Y-torsionfree and Z-torsionfree, we have that the non-zero localization maps appearing in the definition of the coboundary maps are all injective.

If $s = (s_Y, s_Z) \in \mathscr{C}^0(M)$, then $d_0(s) = (0, s_Z - s_Y, s_Y - s_Z, 0) \in \mathscr{C}^1(M)$. Thus $Q_{\kappa_+}(M) \cong \mathbf{H}^0(\pi(M)) \cong \check{H}^0(\tilde{M}) = \operatorname{Ker} d_0 = Q_Y(M) \cap Q_Z(M)$ (within $Q_{YZ}(M)$).

The image of d_0 consists of all four-tuples $\{(0, \xi, -\xi, 0) \mid \xi \in Q_Y(M) + Q_Z(M)\}$. The map d_1 takes $s = (s_{YY}, s_{YZ}, s_{ZY}, s_{ZZ}) \in \mathscr{C}^1(M)$ to $(s_{YY}, s_{YY}, s_{ZY} - s_{YY} + s_{YZ}, s_{ZZ}, s_{YY}, s_{YZ} - s_{ZZ} + s_{ZY}, s_{ZZ}, s_{ZZ}) \in \mathscr{C}^2(M)$. It follows that the kernel of d_1 is equal to $\{(0, \xi, -\xi, 0) \mid \xi \in Q_{YZ}(M)\}$. Whence

$$\check{H}^1(\tilde{M}) = Q_{YZ}(M)/(Q_Y(M) + Q_Z(M)).$$

In the same way, one could prove that $\check{H}^2(\tilde{M}) = 0$.

Since H is Artin−Schelter regular, its cohomology groups are well known (cf. [2]): $\mathbf{H}^i(\pi(H)) = 0$ if $i \notin \{0, 2\}$, $\mathbf{H}^0(\pi(H)) = H$, and $\mathbf{H}^2(\pi(H)) \cong H^*$ [3] where H^* denotes the linear dual $\operatorname{Hom}_k(H, k)$. Combined with the resolution

$$0 \to H[-n-2] \to H[-2] \oplus H[-n-1] \to P \to 0$$

this yields that $\mathbf{H}^0(\pi(P)) = P$. Writing down the long exact cohomology sequence corresponding to $0 \to P \to H \to M \to 0$ yields that $\mathbf{H}^1(\pi(P)) \cong \mathbf{H}^0(\pi(M))/M \cong (Q_Y(M) \cap Q_Z(M))/M$.

A long but easy calculation entails a complete description of the latter module. If $0 \le p < n$, then a basis of $\mathbf{H}^1(\pi(P))_p$ is given by the elements

$$\frac{x^i}{Z^{i-p}} = (i-1-n)\ldots(p-n)\frac{x^p z^{i-p}}{Y^{i-p}}, \qquad p+1 \le i \le n.$$

If $-n \le p < 0$, then $\mathbf{H}^1(\pi(P))_p$ has a basis consisting of

$$\frac{x^i}{Z^{i-p}} = (i-1-n)\ldots(-n)\frac{z^{i+p}}{Y^i}, \qquad -p \le i \le n.$$

Moreover, $\mathbf{H}^1(\pi(P))_p = 0$ for all other values of p.

Note that both $\mathbf{H}^1(\pi(P))_{-1}$ and $\mathbf{H}^1(\pi(P))_0$ are n-dimensional and that it is easy to calculate the maps (from the former to the latter) induced by right multiplication with X, Y, and Z. In particular, the matrices of these maps with respect to the bases displayed above are precisely the ones exhibited in [4].

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